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# Rent extraction with securities plus cash\*

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## Abstract

Auctions employing steeper securities generate greater revenues when bidders have equal opportunity costs. However, when opportunity costs rise sufficiently quickly with valuations, security bids decrease in NPV and steeper securities reduce seller revenues. We show that when such adverse selection obtains, using combinations of securities with differing steepness can generate higher revenues than using securities of the same steepness. We determine the optimal combination of cash plus equity; identify a novel way of implementing the optimal mechanism via decreasing royalty rates; establish the robustness of the mechanism; and identify when auction designs combining cash with steeper-than-equity securities increase seller revenues.

*Key words:* Security design; Combining securities

*JEL classification:* D44; D82

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Dan Bernhardt declares that he has nothing to disclose.

Tingjun Liu declares that he has nothing to disclose.

# 1 Introduction

In many auctions, bidders pay not with pure cash, but with securities or combinations of cash and securities. For example, Andrade, Mitchell, and Stafford (2001) report that payments for 70% of mergers and acquisitions are either entirely in equity, or feature combinations of equity and cash; and Skrzypacz (2013) highlights how oil and gas lease auctions and timber lease auctions typically feature cash payments plus equity payments in the form of royalties. The distinguishing feature of such security-bid auctions is that the payment’s value is tied to the cash flows the bidder generates (Hansen 1985; Demarzo, Kremer and Skrzypacz 2005). For instance, in a takeover auction, an acquirer’s offer to pay a fraction of the merged firm’s equity has a monetary value that is proportional to the value of the joint firm,  $v$ , which is the sum of the target’s value under the bidder’s control plus the bidder’s standalone value.

DeMarzo, Kremer and Skrzypacz (2005) provide a comprehensive analysis of security-bid auctions. They show that a critical factor determining seller’s revenue is the steepness of the security, which measures the sensitivity of the security’s value to the underlying cash flows. For instance, call options are steeper than equities, which are steeper than cash. Demarzo et al. (2005) find that if bidders have private information about the asset’s value but their opportunity costs of winning (e.g., standalone values in takeover auctions) are identical, then auctions using steeper securities yield a seller greater expected revenues.<sup>1</sup>

In practice, bidders with higher valuations  $v$  may face greater opportunity costs  $x(v)$ , for example reflecting that an acquiring firm with a higher standalone value is more likely to generate higher synergies. So too, in project-rights auctions, a higher valuation may reflect that the resources a bidder must commit if it wins may alternatively be more valuably employed elsewhere. Moreover, these opportunity costs are typically private information. If  $x(v)$  increases in  $v$  sufficiently fast, then an extreme form of adverse selection obtains (Che and Kim 2010): bidding strategies in security-bid auctions are decreasing—bidders with high NPVs bid less because they incur all of their high opportunity costs but only retain a share of their high valuations—so that steeper securities yield lower seller revenues.

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<sup>1</sup>Work on rent extraction and security-bid auctions includes Hansen (1987), Fishman (1988, 1989), Dasgupta and Tsui (2004), Board (2007), Povel and Singh (2010), Kogan and Morgan (2010), Gorbenko and Malenko (2011), Skrzypacz (2013), and Sogo, Bernhardt and Liu (2016). See Burkart and Lee (2015) for security design in takeover settings.

Our paper shows that when adverse selection obtains, auction designs that combine securities with *differing* levels of steepness can yield higher revenues than auction designs that use securities from any single class of steepness, and that the greater the difference in steepness between the securities, the better it is for the seller. Underscoring how combining different classes of securities can help, we establish that when (1) extreme adverse selection arises with equity, and (2)  $x(\cdot)$  is weakly convex, a seller can extract *all* rents—the highest-NPV bidder wins and receives zero rent—by using positive combinations of cash and equity. By contrast, using either pure cash or pure equity would generate strictly less revenues. We solve for the optimal combination of cash and equity, and provide the necessary and sufficient conditions for full extraction via cash and equity. Our cash-plus-equity mechanism is robust: a seller can extract almost all rents when the seller only knows the approximate form of  $x(\cdot)$ . Moreover, when the conditions for full extraction via cash plus equity fail, we show that auction designs combining cash with securities that are steeper than equity can increase seller revenues. Collectively, our findings reveal how the insights of DeMarzo et al. (2005) on the advantages of steeper securities extend to settings with adverse selection when the seller can combine auction design with security design.

The intuition is as follows. When bidding strategies are decreasing in equity auctions, low-NPV bidders may extract more rents with equity payments than high-NPV bidders. By contrast, all bidders value cash in the same way, so that with cash a bidder's rent always rises with his NPV. Starting from pure equity, one can reduce the equity payment and increase the cash payment, so that the differential rent of a low type over a high type falls, crossing zero at some point. We derive conditions on  $x_i(\cdot)$  for all bidders  $i$  under which a mix of equity and nonnegative cash exists that (1) makes the (local) differential rent zero at all values of  $v_i$ , (2) ensures the (global) incentive-compatibility of the mechanism, and (3) selects the highest-NPV bidder whenever this NPV is positive. With such a mix, a seller extracts full rents.

Our result is even stronger than what this discussion may suggest. In equity auctions with decreasing strategies, it may still be that, as in pure cash auctions, high NPV bidder types extract more rents than lower bidder types. Thus, one might conjecture that in such cases, when a seller uses positive combinations of cash and equity, high types would earn strictly more rents than low types, implying that full extraction is impossible. This conjecture is false: the optimal mix extracts rent more efficiently than do standard formats. In the full-

extraction mechanism, a seller tailors the menu of payment combinations so that (1) higher types choose a less steep mix that requires them to give up a smaller equity claim to their higher expected revenues in return for a higher cash payment; while (2) lower types choose steeper mixes because they care less about ceding greater equity claims to lower revenues, and more about the cash payment. By properly choosing the rate of variation in the steepness of the mix, a seller can reduce the differential rents between high and low types to zero.

We extend the analysis to a more general setting that relaxes restrictions on  $x_i(v_i)$ —adverse selection need not be extreme—and allows for negative cash payments. We derive the necessary and sufficient conditions for full extraction via combinations of equity and cash. Global incentive compatibility of a mechanism that extracts full rents via cash-plus-equity requires that  $x_i(\cdot)$  be weakly convex. When this condition holds, our mechanism can be implemented simply by having bidders bid cash with a royalty-rate that *declines* with the cash bid. Moreover, we establish that a seller can extract almost all surplus if  $x_i(v_i)$  is ‘almost’ convex, or if there is limited uncertainty about the value of  $x_i$  conditional on  $v_i$ . As noted at the outset, auctions involving positive combinations of cash and security payments are common in oil, gas and timber lease auctions, and (positive) combinations of cash and equity are often used in mergers and acquisitions. Our analysis provides foundations for such mixes.

Our mechanism has the desirable feature that it is dominant strategy incentive compatible. A bidder’s winning payment depends only on his own report; and if a bidder misreports and wins, his profit is negative—and strictly negative if  $x_i(v_i)$  is strictly convex. Thus, the mechanism extracts full rents even from a single bidder, absent the competition that Crémer and McLean (1985, 1988) exploit when bidders have correlated signals.

We then go beyond equity-plus-cash designs to let sellers use securities from an ordered set (Demarzo et al., 2005). Focusing on equity plus cash is not without loss of generality: if  $x_i(v_i)$  is not convex, a seller cannot extract all rents with equity and cash, but can do so using cash and steeper-than-equity securities if  $x_i(v_i)$  is not too concave. Intuitively, the security payment and opportunity cost collectively comprise a bidder’s costs, and convexity in a security’s expected payoff can compensate for a lack of convexity in  $x_i(\cdot)$ . Moreover, if the bidding strategy would be decreasing in a pure security bid auction, a seller can extract all rents by combining securities and positive cash, generating more revenues than with either the security or cash alone. Collectively, our findings indicate that when adverse selection arises,

a seller is better off mixing the least-steep security (cash) with the steepest security. The mix creates wider variation in the resulting steepness, which helps reduce the differential rents of higher types over lower types and ensure the global incentive compatibility of the mechanism.

Ekmekci, Kos, and Vohra (2016) consider the sale of a firm to a single buyer who is privately informed about cash flows and the benefits of control. The seller can offer a menu of cash-equity mixtures, and the bidder must obtain over 50% of the target to gain control. They provide sufficient conditions for the optimal mechanism to take the form of a take-it or leave-it offer for either the smallest stake in the firm that facilitates transfer of control, or for all shares of the firm. By contrast, we show how combinations of equity and cash—where the resulting steepness varies with bidder type—improves seller revenues, and how auction designs that use steeper-than-equity securities can help further.

## 2 The model

A group of  $n \geq 1$  risk-neutral bidders competes to acquire an indivisible asset. Each bidder  $i$  has a private type  $v_i \in [\underline{v}_i, \bar{v}_i]$ , where  $0 < \underline{v}_i < \bar{v}_i < \infty$ , and  $v_i$  is the expected revenues generated by the asset if bidder  $i$  wins. That is, the asset yields a stochastic cash flow with an expected value of  $v_i$  under the control of that type of bidder  $i$ . We assume that  $v_i$  is distributed according to a continuous and strictly positive density,  $f_i$ , and that valuations are independently, but not necessarily identically, distributed across bidders. We denote the joint density of the valuations of bidders other than  $i$  by  $f_{-i}(\mathbf{v}_{-i}) \equiv \prod_{k \neq i} f_k(v_k)$ , where  $\mathbf{v}_{-i} \equiv (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ . We normalize the asset's value to the seller, if retained, to 0.

With equity payments, only the expected cash flow is relevant, but when we allow for more general security payments, their values depend on the details of the cash flow distribution, requiring us to impose more structure. We assume that when controlled by type  $v_i$  of bidder  $i$ , the asset generates a stochastic cash flow  $y_i \sim g(y_i|v_i)$  with full support on  $(0, \infty)$ , where the family of probability density functions  $\{g(\cdot|v_i)\}$  has the *strict monotone likelihood ratio property (sMLRP)*:  $g(y|v_i)/g(y|v'_i)$  increases in  $y$  for  $v_i > v'_i$ , i.e., higher signals are good news.

Bidder  $i$  has an opportunity cost  $x_i(v_i)$  that varies with his valuation. We assume that  $x_i(v_i)$  is a continuous and twice-differentiable function of  $v_i \in [\underline{v}_i, \bar{v}_i]$ . Thus, the expected



NPV (value added) of the asset if  $i$  wins is  $v_i - x_i(v_i)$ . Define  $S_i$  to be the set of  $v_i$  for which there exist realizations of other bidders' types such that selling the asset to bidder  $i$  maximizes expected social surplus, i.e.,

$$S_i = \{v_i | v_i - x_i(v_i) \geq 0 \text{ and for all } j \neq i, v_i - x_i(v_i) \geq \min_{v_j} \{v_j - x_j(v_j)\}\}.$$

We assume that  $S_i$  is non-empty for all  $i$ , which rules out uninteresting cases where selling to a particular bidder is never socially optimal.

## 2.1 Mechanisms using equity plus (possibly negative) cash

We first analyze mechanisms in which the winner pays with combinations of equity and cash. Without loss of generality, we consider direct-revelation mechanisms. Let  $W_i(z_i, \mathbf{v}_{-i})$  be the probability bidder  $i$  wins when he reports being type  $z_i$  and other bidders report  $\mathbf{v}_{-i}$ ; let

$$Q_i(z_i, \mathbf{v}_{-i}) \in [0, 1] \tag{1}$$

be the equity share that bidder  $i$  retains contingent on winning; and let  $M_i(z_i, \mathbf{v}_{-i}) \in (-\infty, \infty)$  be the associated cash payment. We define  $G_i(z_i)$  to be the probability that bidder  $i$  wins when he reports  $z_i$  and all other bidders report truthfully:

$$G_i(z_i) \equiv \int W_i(z_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}. \tag{2}$$

Similarly, we define  $q_i(z_i)$  to be the expected equity share that bidder  $i$  retains conditional on winning by reporting that he has valuation  $z_i$  when all others report truthfully,

$$q_i(z_i) G_i(z_i) \equiv \int Q_i(z_i, \mathbf{v}_{-i}) W_i(z_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}, \tag{3}$$

and define  $\omega_i$  to be the unconditional expected cash payment,

$$\omega_i(z_i) \equiv \int M_i(z_i, \mathbf{v}_{-i}) W_i(z_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}. \tag{4}$$

We define  $h_i(v_i, z_i)$  to be bidder  $i$ 's expected profit when he has valuation  $v_i$  but reports

$z_i$ , and all other bidders report truthfully:

$$\begin{aligned} h_i(v_i, z_i) &\equiv \int [v_i Q_i(z_i, \mathbf{v}_{-i}) - x_i(v_i)] W_i(z_i, \mathbf{v}_{-i}) f_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i} - \omega_i(z_i) \\ &= [v_i q_i(z_i) - x_i(v_i)] G_i(z_i) - \omega_i(z_i). \end{aligned} \quad (5)$$

Bidder  $i$ 's equilibrium expected profit is  $h_i(v_i, v_i)$ . Incentive compatibility requires

$$h_i(v_i, v_i) = \max_{z_i} h_i(v_i, z_i). \quad (6)$$

By the envelope theorem,

$$h_i(v_i, v_i) = h_i(\underline{v}_i, \underline{v}_i) + \int_{\underline{v}_i}^{v_i} \left( q_i(t) - \frac{dx_i(t)}{dt} \right) G_i(t) dt$$

and

$$\frac{dh_i(v_i, v_i)}{dv_i} = \left( q_i(v_i) - \frac{dx_i(v_i)}{dv_i} \right) G_i(v_i). \quad (7)$$

We next identify necessary and sufficient conditions for a mechanism using combinations of equity and cash to exist that extracts full rents—the highest-NPV bidder wins if his NPV is positive and a seller retains the asset otherwise, and all bidder types earn zero rent. If such a mechanism exists, then

$$h_i(v_i, v_i) = 0 \quad \text{for all } v_i, \quad (8)$$

and  $G_i(v_i) > 0$  if and only if  $v_i \in S_i$ . For  $z_i \in S_i$ , divide both sides of (5) by  $G_i(z_i)$  to define

$$\hat{h}_i(v_i, z_i) \equiv \frac{h_i(v_i, z_i)}{G_i(z_i)} = v_i q_i(z_i) - x_i(v_i) - \frac{\omega_i(z_i)}{G_i(z_i)}, \quad (9)$$

which is bidder  $i$ 's expected profit conditional on winning when he has valuation  $v_i$  but reports  $z_i$ , and all other bidders report truthfully. Equations (8) and (9) yield

$$\hat{h}_i(v_i, v_i) = 0, \quad \text{for all } v_i \in S_i. \quad (10)$$

Equation (8) and the incentive compatibility condition (6) yield

$$h_i(v_i, z_i) \leq 0, \quad \text{for all } z_i \neq v_i, \quad (11)$$

and hence

$$\hat{h}_i(v_i, z_i) \leq 0, \quad \text{for all } z_i \in S_i.$$

Thus, for all  $v_i \in S_i$ , we have

$$\hat{h}_i(v_i, v_i) = \max_{z_i \in S_i} \hat{h}_i(v_i, z_i) = \max_{z_i \in S_i} \left( v_i q_i(z_i) - x_i(v_i) - \frac{\omega_i(z_i)}{G_i(z_i)} \right).$$

Adding  $x_i(v_i)$  to both sides yields that, for all  $v_i \in S_i$ ,

$$\hat{h}_i(v_i, v_i) + x_i(v_i) = \max_{z_i \in S_i} \left( v_i q_i(z_i) - \frac{\omega_i(z_i)}{G_i(z_i)} \right). \quad (12)$$

By (12),  $\hat{h}_i(v_i, v_i) + x_i(v_i)$  is the maximum of a family of affine functions and hence is weakly convex for  $v_i \in S_i$ . Because  $\hat{h}_i(v_i, v_i) = 0$ ,  $x_i(v_i)$  itself must be weakly convex for  $v_i \in S_i$ .<sup>2</sup> The necessity of the convexity of  $x_i(v_i)$  reflects that the opportunity costs of higher types must rise at least linearly with  $v_i$ , else a higher  $v_i$  type can extract strictly positive rents from an equity-cash combination designed to extract all rents from lower valuation types. Thus, a necessary condition for full extraction is that  $x_i(v_i)$  be weakly convex:

**Lemma 1** *If  $x_i(v_i)$  is not weakly convex over  $v_i \in S_i$  for all  $i$ , then combinations of equity plus cash cannot extract all surplus, even if the cash component is allowed to be negative.*

We now provide all necessary conditions for there to exist a mechanism employing combinations of equity and cash that extracts full rents. These conditions are also sufficient:

**Theorem 1** *A mechanism using combinations of equity and (possibly negative) cash exists that extracts full rents if and only if for all bidders  $i$ , (i)  $x_i(v_i)$  is weakly convex in  $v_i \in S_i$ , (ii)  $0 \leq \frac{dx_i(v_i)}{dv_i} \leq 1$  for  $v_i \in S_i$ , and (iii) for  $v_i \notin S_i$ ,  $x_i(v_i) \geq x_i(z_i) + x'_i(z_i)(v_i - z_i)$  for  $z_i \in S_i$ .*

In the special case that bidders are ex-ante identical and surplus is always positive,  $v_i \in S_i$  for all  $v_i$ , rendering condition (iii) irrelevant. Then Theorem 1 takes a simpler form:

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<sup>2</sup>If  $S_i$  is not connected, then the requirement translates to requiring that a weakly convex function of  $v_i$  go through  $(v_i, x_i(v_i))$  for all  $v_i \in S_i$ . The assumptions that  $x_i(v_i)$  is twice differentiable and  $S_i$  is non-empty imply that  $S_i$  must be connected if (i) and (ii) of Theorem 1 hold. However, with a finite type space, as in Example 1,  $S_i$  need not be connected.

**Corollary 1** *When bidders are ex-ante identical with  $v - x(v) > 0$  for all  $v \in [\underline{v}, \bar{v}]$ , a mechanism using combinations of equity and (possibly negative) cash exists that extracts full rents if and only if  $x(v)$  is weakly convex in  $v$  and  $0 \leq \frac{dx(v)}{dv} \leq 1$  for  $v \in [\underline{v}, \bar{v}]$ .*

Returning to the more general case of (possibly) heterogeneous bidders, we next establish some salient properties of the optimal mechanism.

**Corollary 2** *When conditions (i), (ii) and (iii) in Theorem 1 hold, the following dominant strategy incentive compatible mechanism extracts full rents:*

- The winning rule is  $W_i(v_i, \mathbf{v}_{-i}) = \begin{cases} 1 & \text{if } v_i \in S_i \text{ and } v_i - x_i(v_i) > \max_{j \neq i} v_j - x_j(v_j) \\ 0 & \text{otherwise} \end{cases}$ ,  
and ties are broken arbitrarily.
- The winning bidder retains equity share

$$Q_i(v_i, \mathbf{v}_{-i}) = \frac{dx_i(v_i)}{dv_i}, \quad (13)$$

and makes cash payment

$$M_i(v_i, \mathbf{v}_{-i}) = v_i \frac{dx_i(v_i)}{dv_i} - x_i(v_i). \quad (14)$$

**Proof of Theorem 1 and Corollary 2:** We first prove that when the conditions in Theorem 1 hold, the mechanism in Corollary 2 extracts full rents. If bidding is truthful, then substituting for  $M_i(v_i, \mathbf{v}_{-i})$  and  $q_i(v_i)$  in equation (5) reveals that a bidder's equilibrium payoff is zero, regardless of his valuation, i.e.,

$$[q_i(v_i)v_i - x_i(v_i) - M_i(v_i, \mathbf{v}_{-i})]G_i(v_i) = 0.$$

To see that truthful bidding is an equilibrium, suppose type  $v_i$  bids as if he is type  $z_i$ . If  $z_i \notin S_i$ , the bidder would lose, so the deviation is not optimal. If  $z_i \in S_i$ , then by (13) and (14),

$$q_i(z_i) = \frac{dx_i(z_i)}{dz_i} \quad (15)$$

and

$$\omega_i(z_i) = \left[ z_i \frac{dx_i(z_i)}{dz_i} - x_i(z_i) \right] G_i(z_i). \quad (16)$$

Then by (5), we have

$$\begin{aligned} h_i(v_i, z_i) &= \left[ v_i \frac{dx_i(z_i)}{dz_i} - x_i(v_i) \right] G_i(z_i) - \left[ z_i \frac{dx_i(z_i)}{dz_i} - x_i(z_i) \right] G_i(z_i) \\ &= \left[ (v_i - z_i) \frac{dx_i(z_i)}{dz_i} - x_i(v_i) + x_i(z_i) \right] G_i(z_i) \leq 0, \end{aligned} \quad (17)$$

where the inequality (17) holds for all  $v_i \in [\underline{v}_i, \bar{v}_i]$ : if  $v_i \in S_i$ , the inequality follows because the weak convexity of  $x_i(\cdot)$  (in condition (i) of Theorem 1) implies that  $(v_i - z_i) \frac{dx_i(z_i)}{dz_i} - x_i(v_i) + x_i(z_i) \leq 0$  regardless of whether  $v_i \geq z_i$  or  $v_i < z_i$ ; if  $v_i \notin S_i$ , the inequality follows from part (iii) of Theorem 1. Thus, deviation is not profitable for all bidder types. Hence, Corollary 2 and the “if” part of Theorem 1 are established.

We next prove the “only if” part of Theorem 1 by assuming that a full-extraction mechanism exists. The necessity of condition (i) was proved in the text. Further, for all  $v_i \in S_i$ , full extraction implies  $G_i(v_i) > 0$  and  $\frac{dh_i(v_i, v_i)}{dv_i} = 0$ , which, by (7), yields  $\frac{dx_i(v_i)}{dv_i} = q_i(v_i)$ . By (1), condition (ii) follows. To prove that (iii) must hold, note that full extraction implies that for all types  $v_i \notin S_i$ , bidding as if the bidder’s type is  $z \in S_i$  must render a non-positive profit; by (17) and  $G_i(z_i) > 0$ , we have  $(v_i - z_i) \frac{dx_i(z_i)}{dz_i} - x_i(v_i) + x_i(z_i) \leq 0$ , establishing (iii). ■

Condition (i) of Theorem 1 ensures that no type in  $S_i$ —the set of types for which the social surplus from selling the asset to them is positive—wants to mimic another type in  $S_i$ . Condition (ii) reflects the constraint that the retained equity share must be between zero and one. Specifically,  $\frac{dx_i(v_i)}{dv_i} \geq 0$  says that bidder types that expect to generate higher revenues also face higher opportunity costs. If this condition does not hold, then a bidder who expects to generate higher revenues can extract strictly positive surplus from an equity-cash combination designed to extract all surplus from any lower  $v_i$  type. The requirement that  $\frac{dx_i(v_i)}{dv_i} \leq 1$  means that it is socially more efficient to allocate the asset to a higher  $v_i$  type even though that type also has higher opportunity costs. If this condition does not hold, then any equity cash combination designed to extract all rents from a  $v_i$  type for which  $\frac{dx_i(v_i)}{dv_i} > 1$  will provide strictly positive rents to marginally lower types. Condition (ii) is mild—in practice, it is likely to be satisfied. In particular, Condition (ii) is equivalent to requiring that the synergy a bidder can generate increases in its standalone value (via the relation that synergy equals  $v - x$ ). Condition (iii) ensures that types not in  $S_i$  to whom the asset should not be sold do not want to mimic a type in  $S_i$  to whom the seller might want to sell the asset.

Condition (iii) is implied if condition (i) also holds for  $v_i \notin S_i$ : Condition (iii) is weaker than requiring (i) to hold for  $v_i \notin S_i$  (given that (i) is required for  $v_i \in S_i$ ). The difference in the restrictiveness of the two conditions reflects that for any type *in*  $S_i$ , one must ensure that no other type wants to mimic it. By contrast, for types not in  $S_i$ , such a requirement is unnecessary because mimicking a type that is not in  $S_i$  always loses.

**Corollary 3** *In the full extraction mechanism (Corollary 2), higher types pay with flatter securities: the equity share paid falls with type and the cash payment rises with type, i.e.,*

$$\frac{d}{dx}(1 - Q_i(v_i)) = -\frac{d^2 x_i(v_i)}{d^2 v_i} \leq 0 \quad \text{and} \quad \frac{dM_i(v_i)}{dv_i} = v_i \frac{d^2 x_i(v_i)}{d^2 v_i} \geq 0.$$

This corollary reveals the general principle underlying the gains from mixing cash and equity: a seller can tailor the menu of cash-equity payment combinations so that (1) higher types that expect to generate higher revenues from the project (but incur higher opportunity costs) choose a less steep mix that requires them to give up a smaller equity claim to those revenues in return for a higher cash payment;<sup>3</sup> while (2) lower types choose steeper mixes because they care less about ceding greater equity claims to lower revenues, and more about the cash payment. The net effect is to reduce the differential rents between high and low types.

The rent-extracting ability of the cash-equity mix (Corollary 2) and Corollary 3 can reconcile findings by Andrade et al. (2001) for mergers and acquisitions that (1) a target (seller) appears to grasp all synergy gains—the acquirer does not earn a positive abnormal return—and (2) the target’s abnormal return is higher when an acquirer’s payment has a larger cash component (**revealing that its type is larger**).<sup>4</sup>

The mechanism in Corollary 2 has the desirable feature that it is dominant strategy incentive compatible. A bidder’s winning payment depends only on his own report, and not those of other bidders; and if a bidder misreports and wins, his profit is non-positive. Thus, the mechanism extracts full rents even from a single bidder absent any competition,

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<sup>3</sup>In contrast, in equity auctions with constant opportunity costs, a higher type always gives up a *larger* claim to outbid lower types.

<sup>4</sup>Our mechanism applies to settings in which the seller has the bargaining power. The seller optimally offers a menu of payment choices for bidders, in which higher cash payments are associated with reduced equity claims. Higher valuation bidders select mixtures with higher cash payments, and the seller awards the asset to the bidder who offers the highest cash payment. Example 2 illustrates how the mechanism can be implemented in an auction.

and bidders would not deviate even in complete information setting (when there are multiple bidders). Moreover, if  $x_i(v_i)$  is strictly convex over  $[\underline{x}_i, \bar{x}_i]$ , the equilibrium enforces itself in a strict sense: bidder  $i$  would receive a strictly negative expected profit if he deviated and won.

## 2.2 Mechanisms using equity plus positive cash

The cash payments specified in Corollary 2 needed to extract all rents could be negative. In practice, there may be concerns with negative cash payments (DeMarzo, Kremer and Skrzypacz 2005). Empirically, mixtures of securities with positive cash payments are often observed, for example in mergers and acquisitions, but negative cash payments are not. We now provide the additional necessary and sufficient condition for the optimal mechanism to specify that the winning bidder make a strictly positive cash payment ( $\omega_i(v_i) > 0$  for all  $v_i \in S_i$ ):

**Corollary 4** *Suppose that conditions (i), (ii) and (iii) in Theorem 1 hold. Then, if*

$$\frac{d}{dv_i} \frac{x_i(v_i)}{v_i} > 0 \quad \text{for all } i, \quad (18)$$

*the mechanism specified in Corollary 2 uses strictly positive combinations of equity and cash and extracts all rents, generating strictly higher expected revenues than either pure equity or pure cash. If these conditions do not hold then no mechanism can extract full rents using combinations of equity and strictly positive cash.*

To see that the result holds, observe that when the conditions in Theorem 1 and (18) hold, the mechanism specified in Corollary 2 extracts full rents. Further, by (18), the cash component in (14),  $v_i \frac{dx_i(v_i)}{dv_i} - x_i(v_i)$ , is strictly positive. Moreover, a seller must use strictly positive amounts of both cash and equity to extract full rents. To see the converse part of Corollary 4, suppose a full-extraction mechanism exists that employs strictly positive cash. For all  $v_i \in S_i$ , full extraction implies  $G_i(v_i) > 0$  and  $\frac{dh_i(v_i, v_i)}{dv_i} = 0$ , which yields  $\frac{dx_i(v_i)}{dv_i} = q_i(v_i)$  by (7). By  $h_i(v_i, v_i) = 0$  and (5),

$$\omega_i(v_i) = \left( \frac{dx_i(v_i)}{dv_i} v_i - x_i(v_i) \right) G_i(v_i),$$

which, by  $\omega_i(v_i) > 0$  as implied by cash component being positive, yields

$$\frac{dx_i(v_i)}{dv_i}v_i - x_i(v_i) > 0,$$

which yields (18).

Equation (18), which is equivalent to the elasticity condition  $\frac{d \ln x_i}{d \ln v_i} > 1$ , is the necessary and sufficient condition for bidding strategies in second-price equity auctions to be decreasing (Che and Kim 2010). This follows because the bidding strategy in second-price equity auctions takes the form  $1 - \frac{x_i}{v_i}$ .<sup>5</sup> In Corollary 4, it ensures that cash transfers are positive. Importantly, this condition is unrelated to the *scale* of opportunity costs relative to valuations. For example, it holds for all  $x_i = \alpha v^\beta$  for  $\beta \geq 1$  (when  $\alpha$  is small enough that condition (ii) holds).

The following example illustrates that a higher-NPV type can earn higher rents in an equity auction than a lower type, and hence earn strictly positive rents in pure equity and pure cash auctions, but nonetheless earns zero rents when equity and cash are combined optimally.

**Example 1.** Suppose there are two ex-ante identical bidders whose expected valuations can take on one of three values,  $v_i \in \{1, 2, 5\}$ , with associated probabilities, 0.1, 0.1 and 0.8, and associated opportunity costs of  $x(1) = 0.1$ ,  $x(2) = 0.3$ , and  $x(5) = 3$ .

Consider second price equity auctions in which ties are broken randomly. A bidder with valuation  $v$  then bids to retain an equity share of  $\frac{x(v)}{v}$ . The parameterization implies that strategies decrease with a bidder's valuation. A bidder with low valuation  $v = 1$  wins when the rival bidder has valuation 2 and bids to retain share  $\frac{0.3}{2}$ ; he wins with when the rival bidder has valuation 5 and bids to retain share  $\frac{3}{5}$ ; and his profits are zero when they have the same valuation. Thus, type  $v = 1$ 's expected profit is

$$\pi_{v=1} = 0.1 \times \left( \frac{0.3}{2} \times 1 - 0.1 \right) + 0.8 \times \left( \frac{3}{5} \times 1 - 0.1 \right) = \frac{0.01}{2} + 0.4 = 0.405.$$

A bidder with valuation  $v = 2$  earns expected profit of

$$\pi_{v=2} = 0.80 \times \left( \frac{3}{5} \times 2 - 0.3 \right) = 0.72 > \pi_{v=1}.$$

This example has a discrete set of types corresponding to three points in the  $v$  vs  $x$  plane.

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<sup>5</sup>This reflects that a bidder who retains an equity stake of  $\frac{x_i}{v_i}$ , just breaks even,  $\frac{x_i}{v_i}v_i - x_i = 0$ .



Condition (i) of Theorem 1 for full extraction demands that a weakly convex function  $x(v)$  go through these three points, and there is an infinite number of such functions, giving rise to an infinite number of ways to implement the mechanism in Corollary 4. Observe that a straight line joining points 1 and 2 has a slope of  $\frac{0.3-0.1}{2-1} = 0.2$ ; and a straight line joining points 2 and 3 has a slope of  $\frac{3-0.3}{5-2} = 0.9$ , which exceeds 0.2. Denote by  $k(1)$ ,  $k(2)$  and  $k(5)$  the slope  $\frac{dx_i(v_i)}{dv_i}$  that determines the equity retention (see (13)) at these 3 points. Then the only constraints imposed by the convexity requirement on  $x(v)$  are:  $k(1) \in [0, 0.2]$ ,  $k(2) \in [0.2, 0.9]$ , and  $k(3) \in [0.9, 1]$ . Below, we (arbitrarily) pick  $k(1) = 0.15$ ,  $k(2) = 0.55$ , and  $k(5) = 0.95$ . The mechanism in Corollary 4 yields:

- If the reported value is  $v = 1$ , pay fraction 0.85 of equity and 0.05 in cash.
- If the reported value is  $v = 2$ , pay fraction 0.45 of equity and 0.8 in cash.
- If the reported value is  $v = 5$ , pay fraction 0.05 of equity and 1.75 in cash.
- The highest reported type wins and ties are broken randomly.

Substitution into a bidder's expected profits reveals that truth telling results in zero payoffs and deviating results in strictly negative payoffs. In this example, in an equity auction, a higher-NPV type ( $v = 2$ ) earns higher rents than a lower NPV type ( $v = 1$ ). Thus, both in pure equity auctions *and* in pure cash auctions,  $v = 2$  earns strictly more rents than  $v = 1$ . Nonetheless the optimal strictly positive mixture of equity and cash extracts rents more efficiently, so that both  $v = 2$  and  $v = 1$  earn zero rents.

**Implementation.** The full extraction mechanism can be implemented via cash plus a *decreasing* royalty-rate as follows. By Corollary 3, the equity share paid falls with  $v_i$  and the cash payment rises with  $v_i$ . Therefore, the equity share paid can be expressed as a strictly decreasing function of the cash payment. This means that the full extraction mechanism can be implemented by having bidders bid in cash, and setting a reserve price that equals the cash payment in (14) for zero-NPV type (or for type  $\underline{v}_i$  if all types have positive NPVs). The bidder with the highest bid exceeding the reserve wins (assuming ex-ante identical bidders; the approach generalizes) and pays his own bid. However, for each cash bid made, a bidder must pay an additional equity share that is uniquely determined by his cash bid via

the function determined above. As with the mechanism in Corollary 2, this implementation has the desirable feature that it is dominant strategy incentive compatible; and if  $x_i(v_i)$  is strictly convex for all  $i$ , the equilibrium is unique.

**Example 2.** Let there be  $n$  ex-ante identical bidders with expected valuation  $v$  distributed on  $[2, 3]$  and associated opportunity costs  $x(v) = 0.3 + 0.4v + 0.1v^2$ .

Let  $c$  be the cash bid and  $\alpha(c)$  be the associated royalty rate paid. We solve for the  $\alpha(c)$  that extracts all rents. By (14),  $c$  relates to  $v$  via  $c = v \frac{dx(v)}{dv} - x(v) = 0.1v^2 + 0.3$ . Solving for  $v$  in terms of  $c$  yields  $v = \sqrt{10c + 3}$ . By (13),  $1 - \alpha = \frac{dx(v)}{dv} = 0.4 + 0.2v$ . Solving for  $\alpha$  in terms of  $c$  yields:

$$\alpha(c) = 0.6 - 0.2\sqrt{10c + 3}. \quad (19)$$

Now, consider a first-price auction in which the highest cash bid wins and the winner pays his cash bid  $c$  plus the royalty rate given by (19). Denote the expected profit of a type  $v$  bidder conditional on winning by  $\pi(c; v)$ . Then

$$\pi(c; v) = (1 - \alpha(c))v - c - x(v) = 0.2\sqrt{10c + 3}v - c - 0.3 - 0.1v^2.$$

$\pi(c; v)$  is concave in  $c$ . It is straightforward to show (via  $\frac{d\pi}{dc} = v/\sqrt{10c + 3} - 1$ ) that for all  $v$ , the maximum value of  $\pi(c; v)$  is zero, which obtains at

$$c = 0.1v^2 - 0.3, \quad (20)$$

and that  $\pi(c; v) < 0$  for all other values of  $c$ . Because a bidder's losing profit is zero, bidding according to (20) is optimal regardless of how other bidders bid. In equilibrium, all bidders receive zero rent. Because (20) strictly increases in  $v$ , the highest valuation bidder wins and the NPV is strictly positive: the seller extracts full rents.

This example highlights another novel feature of our implementation: in a first-price auction using combinations of cash with a royalty rate that declines in the cash bid has an equilibrium in which bidders employ dominant strategies. By contrast, if bidders pay with either pure cash or pure equity, a first-price auction does not have a dominant strategy equilibrium (even if bidders have type-dependent opportunity costs).

**Robustness.** Our mechanism is robust to Samuelson's (1987) concerns about mechanisms

that extract all surplus via contingent payments. First, the equilibrium strictly enforces itself if  $x_i(v_i)$  is strictly convex: bidder  $i$  expects strictly negative profits from deviating and winning. This makes selection of the ‘right’ bidder robust to small noise. Second, a winning bidder almost never gives up all equity, mitigating moral hazard concerns.

We now establish a third robustness property, deriving a lower bound on seller revenues when full extraction is impossible via combinations of cash and equity. We show that if  $x_i(v_i)$  is not quite weakly convex, or there is slight uncertainty about the value of  $x_i$  conditional on  $v_i$ , combining securities of differing degrees of steepness can still extract almost all surplus.

For simplicity, we consider a single bidder and assume that it is socially optimal to sell to any bidder type, i.e.,  $v - x(v) > 0$  for all  $v$ . Thus, the maximum social welfare gain is

$$\Pi \equiv \int_{\underline{v}}^{\bar{v}} (v - x(v)) f(v) dv.$$

**Proposition 1** *Let  $y(v)$  be a weakly convex function over  $v \in [\underline{v}, \bar{v}]$  with  $0 \leq \frac{dy(v)}{dv} \leq 1$  for all  $v \in [\underline{v}, \bar{v}]$ . Then if the opportunity cost function is  $x(v)$ , an equity-plus-cash mechanism generates expected revenue of at least*

$$\Pi - \sup_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')) + \inf_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')).$$

*In this mechanism, the bidder always wins the asset,  $W(v) = 1$ , retains equity share*

$$Q(v) = \frac{dy(v)}{dv}, \tag{21}$$

*and makes cash payment*

$$M(v) = v \frac{dy(v)}{dv} - y(v) - \sup_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')). \tag{22}$$

**Proof:** See appendix. ■

Intuitively, for any  $y(v)$  that satisfies the conditions in Proposition 1, the prescribed mechanism is incentive compatible for any true  $x(v)$ : the bidder’s payoff is (weakly) higher if he truthfully reports his type  $v$  than if he misreports. Furthermore, changing the cash payment by a constant—replacing the cash payment with  $v \frac{dy(v)}{dv} - y(v) - \delta$ , where  $\delta$  is

a constant—only affects individual rationality constraints and not incentive compatibility constraints. Thus, by properly choosing  $\delta$ , both the incentive compatibility and individual rationality constraints can be satisfied. Specifically, setting  $\delta = \sup (x(v) - y(v))$  leaves the bidder type with the highest  $x(v) - y(v)$  zero rents, and it leaves all other bidder types with non-negative rents that do not exceed  $\sup (x(v) - y(v)) - \inf (x(v) - y(v))$ .

Proposition 1 implies that when  $x(v)$  is “close” to being weakly convex, the seller can extract almost all surplus. Moreover, this result holds even if  $x$  does not evolve deterministically with  $v$ , extending to settings where  $x$  and  $v$  are jointly distributed on a two dimensional space, where, for example, conditional on  $v$ ,  $x$  is distributed on support  $[x(v) - \epsilon, x(v) + \epsilon]$ . Thus, even if the seller only knows approximately how  $x$  is related to  $v$ , our mechanism may still deliver close to full extraction, underscoring its robustness.<sup>6</sup>

Cr  mer (1987) shows that when  $x_i$  is constant and common knowledge, a seller can extract full rents by reimbursing a bidder for his opportunity cost, and asking for all equity.<sup>7</sup> Our work extends his insights to settings in which  $x_i$  varies with bidder’s private type. Theorem 1 provides the necessary and sufficient conditions for full extraction in such environments (observe that Cr  mer’s assumption that  $x_i$  is constant satisfies the weak convexity condition of Theorem 1) and Corollary 2 shows how to extract full rents when these conditions are met: a seller should tailor a menu of combinations of equity and cash, where the relative weights vary in such a way that higher types choose less steep mixes and lower types choose steeper mixes.

We now go beyond the equity-plus-cash framework to show that a seller can do better: we identify settings in which full extraction is impossible using equity and cash—the necessary conditions in Theorem 1 do not hold—but a seller can still extract all surplus using cash and a steeper-than-equity security.

## 2.3 Extending to ordered securities other than equity

We next consider settings in which bidders pay with combinations of cash and securities. For simplicity, we focus on a single bidder.<sup>8</sup> To ease presentation, we assume that (i) each

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<sup>6</sup>See Bergemann, Brooks and Morris (2017) for related robustness analyses.

<sup>7</sup>See Fishman (1989), Eckbo, Giammarino and Heinkel (1990), or Gorbenko and Malenko (2018) for papers in which cash-equity compositions arise due to preemption, signaling, or financial constraint considerations.

<sup>8</sup>The mechanism generalizes to allow the seller to extract full rents from multiple bidders, while maintaining dominant strategy incentive compatibility.

bidder type  $v = E[y|v]$  corresponds to a positive net present value, i.e.,  $v - x(v) > 0$ , and (ii) a higher expected cash flow corresponds both to a higher opportunity cost and to a higher NPV, i.e.,  $0 < \frac{dx(v)}{dv} < 1$ , for all  $v$ .

We consider an *ordered set of securities*,  $\{S(s, \cdot); s \in (\underline{s}, \bar{s})\}$ , as introduced in DeMarzo et al. (2005). Each security in such a set is indexed by  $s$ , where for each  $s$ ,  $S(s, \cdot)$  specifies the payoff of the security as a function of the stochastic cash flows that the asset generates. For example, for the set of equity securities, the index  $s$  is the equity fraction and the payoff function is  $S(s, y) = sy$ . More generally, we require (i) for all  $s$  and  $y \in [0, \infty)$ , both  $S(s, y)$  and  $y - S(s, y)$  weakly increase in  $y$ , and  $0 \leq S(s, y) \leq y$ ; and (ii)  $ES(s, v_i) \equiv E[S(s, y_i)|v_i]$ , the expected value of security  $S(s, \cdot)$  conditional on bidder type  $v$ , is strictly increasing in both arguments, differentiable in  $s$ , and twice differentiable in  $v$ . We let  $ES_s(s, v)$  and  $ES_v(s, v)$  denote the partial derivatives with respect to  $s$  and  $v$ , and denote the second derivative with respect to  $v$  by  $ES_{vv}(s, v) \equiv \frac{\partial^2 ES(s, v)}{\partial v^2}$ . To ease analysis, we assume:

**Assumption 1** (i) If  $s_1 > s_2$ , then  $S(s_1, y) \geq S(s_2, y)$  for all  $y \in (0, \infty)$ , and  $S(s_1, y) \geq S(s_2, y)$  is weakly increasing in  $y \in (0, \infty)$ , strictly increasing over a positive measure of  $y$ .

(ii) The lower bound  $\underline{s}$  corresponds to zero payment and the upper bound  $\bar{s}$  corresponds to full payment: for any  $\epsilon > 0$ , there exist  $s^*$  and  $s^{**}$  such that for all  $v$ ,  $ES(s^*, v) < \epsilon$ ,  $ES_v(s^*, v) < \epsilon$ ,  $ES(s^{**}, v) > v - \epsilon$ , and  $ES_v(s^{**}, v) > 1 - \epsilon$ .

Part (i) says that the payment increases in the security index, and the difference in payment widens as the underlying cash flow rises. Part (ii) ensures a wide enough range in the security index. For instance, for call options,  $\underline{s}$  corresponds to infinite strike and  $\bar{s}$  corresponds to zero strike. Both parts hold for standard securities (e.g., debt, equity, or call options).

We consider direct-revelation mechanisms featuring a security plus cash. We first do not restrict the sign of cash. Let  $W(z)$  be the probability that the bidder wins when he reports being type  $z$ ; let  $s(z)$  denote the associated index of the security paid when he reports type  $z$  and wins; and let  $M(z) \in (-\infty, \infty)$  be the cash payment when he reports type  $z$  and wins. Thus, the bidder's expected profit when he has valuation  $v$  but reports  $z$  is

$$h(v, z) = (v - x(v) - ES(s(z), v) - M(z))W(z). \quad (23)$$

If a full-extraction mechanism exists, then  $h(v, v) = 0$  and  $W(v) = 1$  for all  $v$  (recall that all bidder types have positive NPV). Then

$$h(v, z) = v - x(v) - ES(s(z), v) - M(z). \quad (24)$$

Given our full extraction focus, we use (24) in lieu of (23) for  $h$ . Incentive compatibility yields

$$v \in \arg \max_z h(v, z), \quad \text{for all } v. \quad (25)$$

Thus,  $h(v, z) \leq 0$  for all  $z \neq v$ , which, combined with  $h(z, z) = 0$  for all  $z$ , yields

$$z \in \arg \max_v h(v, z), \quad \text{for all } z. \quad (26)$$

Equation (26) holds when a mechanism extracts full rents (or more generally when all bidder types receive the same rent). It is instructive to examine its relation to (25).  $W(v) = 1$ ,  $h(v, v) = 0$  for all  $v_i$ , and the incentive compatibility condition (25) are necessary and sufficient for the mechanism to extract full rents. A similar relation holds for (26):

**Lemma 2** *A mechanism extracts full rents if and only if  $W(v) = 1$  and  $h(v, v) = 0$  for all  $v$  and (26) holds.*

This follows directly: When  $h(v, v) = 0$  for all  $v$ , (25) and (26) imply each other. Lemma 2 simplifies the identification of the conditions for full extraction vis à vis working directly with (25).<sup>9</sup> Equation (24) and the first-order condition for (26),  $\frac{\partial h(v, z)}{\partial v}|_{v=z} = 0$  yield (upon substituting  $v$  for  $z$ ), for all  $v$ :

$$1 - x'(v) - ES_v(s(v), v) = 0. \quad (27)$$

**Lemma 3** *For all  $v \in [\underline{v}, \bar{v}]$ , (27) has a unique solution.*

**Proof:** See appendix. ■

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<sup>9</sup>While the first-order condition of (26) is the same as the standard envelope condition, the second-order condition is more tractable than that of (25).

The second-order condition of (26) gives

$$\frac{\partial^2 h(v, z)}{\partial v^2} \Big|_{v=z} = -\frac{d^2 x(z)}{dz^2} - ES_{vv}(s(z), z) \leq 0,$$

which, upon replacing  $z$  with  $v$ , yields a necessary condition for full extraction:

**Corollary 5** *If combinations of (possibly negative) cash and security from an ordered set extracts full rents, then for all  $v_i \in [\underline{v}, \bar{v}]$ ,*

$$\frac{d^2 x(v)}{dv^2} \geq -ES_{vv}(s(v), v), \text{ where } s(v) \text{ solves (27)}. \quad (28)$$

When the security is equity, the right-hand side of (28) is zero: (28) just says that  $x(v)$  must be weakly convex. By Theorem 1, this is also sufficient for full extraction via equity plus cash. For a general set of ordered securities, matters are more complicated: unlike with equities, where each security can be expressed as a “fraction” of a single base-security, for a general set such as the family of call options indexed by different strikes, securities cannot be expressed as fractions of each other. As a result, (28) is typically not sufficient for global optimality. We next identify sufficient conditions, showing that if  $x(v)$  is concave, but not ‘too’ concave, combinations of steeper-than-equity securities and cash can still extract full rents. To proceed, define

$$k(v) \equiv \min_{z \neq v, z \in [\underline{v}, \bar{v}]} \frac{\int_v^z ES_{vv}(s(v), v) dv}{z - v} \quad (29)$$

for  $v \in [\underline{v}, \bar{v}]$  where  $s(v)$  solves (27). Here,  $k(v)$  is a measure of the minimum average second derivative of expected revenues generated by the security indexed by  $s(v)$ , where the average is taken over points between  $v$  and any other point  $z \in [\underline{v}, \bar{v}]$ . Define

$$K \equiv \min_{v \in [\underline{v}, \bar{v}]} k(v), \quad (30)$$

to be the lower bound on this measure over all  $v \in [\underline{v}, \bar{v}]$ . Because the minimum of averages is no less than the minimum of individual elements,

$$k(v) \geq \min_{v, z \in [\underline{v}, \bar{v}]} ES_{vv}(s(v), z)$$

for all  $v \in [\underline{v}, \bar{v}]$ , which, by (30), yields

$$K \geq \min_{v, z \in [\underline{v}, \bar{v}]} ES_{vv}(s(v), z). \quad (31)$$

Inequality (31) adds to the usefulness of the sufficient conditions for full extraction:

**Theorem 2** *If  $x''(v) + K \geq 0$  for all  $v$ , then a mechanism combining cash and security exists that extracts full rents. The mechanism allocates the asset to the bidder,  $W(v) = 1$ , the security payment  $s(v)$  solves (27), and the cash payment is*

$$M(v) = v - x(v) - ES(s(v), v). \quad (32)$$

**Corollary 6** *Suppose that bidding strategies would be strictly decreasing in a pure second-price security-bid auction. Then the mechanism in Theorem 2 features strictly positive cash payments and generates strictly higher revenues than pure cash or pure securities.*

**Proofs:** See appendix. ■

The condition in Theorem 2 that  $x''(v) \geq -K$  for all  $v$  is a sufficient condition for full extraction via security plus cash. Equity has a linear payoff function, so  $k(v)$  in (29) is zero for all  $v$ , and hence  $K = 0$ . Thus, with equity, the condition just requires  $x''(v) \geq 0$ , i.e., that  $x(v)$  be weakly convex. This is precisely condition (i) in our equity analysis in Theorem 1. When the set of securities is steeper than equities,  $K$  tends to be positive, which relaxes the convexity requirement on  $x(\cdot)$  needed for full extraction. To illustrate, we use the leading example from DeMarzo et al. (2005) for cash flow distributions.

**Assumption 2 (DKS)** *The cash flow  $y = \theta v$ , where  $\theta$  is distributed over  $(0, \infty)$  with a mean of 1 and independent of  $v$ , and  $\log(\theta)$  has a log-concave density function.*

For instance,  $\theta$  could be log-normally distributed. This structure guarantees sMLRP.

To begin, consider a call option with strike price  $v^* > 0$ . Write the expected cash flow for bidder type  $v$  as  $ES(v^*, v)$  (with abuse of notation because  $v^*$  is not a proper index, as a higher strike price corresponds to a lower payoff). Let  $f_\theta(\cdot)$  denote the distribution of  $\theta$ . Then

$$ES(v^*, v) = \int_{\frac{v^*}{v}}^{\infty} (\theta v - v^*) f_\theta(\theta) d\theta,$$



and

$$ES_v(v^*, v) = \int_{\frac{v^*}{v}}^{\infty} \theta f_{\theta}(\theta) d\theta \quad \text{and} \quad ES_{vv}(v^*, v) = \frac{(v^*)^2}{v^3} f_{\theta}\left(\frac{v^*}{v}\right) > 0.$$

Thus,  $K > 0$  (see (31)) for call options. For securities other than call options, we have:

**Lemma 4** *If Assumption 2 holds and the payoff function of each security  $s$  is weakly convex over  $(0, \infty)$ , almost everywhere twice differentiable, but not a straight line, then  $ES_{vv}(s(v), v) > 0$  for all  $v$ , where  $s(v)$  solves (27).*

**Proof:** See appendix. ■

If every security in an ordered set has a weakly convex payoff function as in the lemma, then the set is steeper than equity (Lemma 5, DeMarzo et al. 2005). By Lemma 4 and (31),  $K > 0$ , which, by Theorem 2, relaxes the requirement for full extraction on the convexity of  $x(\cdot)$  relative to equity. Moreover, the steeper is the security, the larger is  $K$ . Intuitively, the security payment and opportunity cost collectively comprise a bidder's costs, and curvature in a security's expected payoff can compensate for a lack of convexity in  $x(\cdot)$ . Example 3 below shows that a dominant strategy incentive compatible mechanism employing combinations of call options and cash can extract all surplus when equity and cash cannot, underscoring how steeper securities help further in the rent extraction.

**Example 3.** Consider a single bidder whose valuations can take on values,  $v \in \{5, 6, 7\}$ , each with strictly positive probability, and associated opportunity costs of  $x(5) = 1$ ,  $x(6) = 1.6$ , and  $x(7) = 1.9$ . For type  $v_i$ , the cash flow  $y$  is uniformly distributed between  $[v - 1, v + 1]$ .

In this example,  $x(v)$  is not weakly convex: the slope of the line joining points 1 and 2 is  $\frac{1.6-1}{6-5} = 0.6$ , and the slope of the line joining points 2 and 3 is  $\frac{1.9-1.6}{7-6} = 0.3 < 0.6$ . Thus, by Theorem 1, full extraction is impossible using equity plus cash, even when cash components can be negative. However, a seller can extract full rents using call options plus cash, where the cash component is positive (because the use of call options results in decreasing bidding strategies). The payments are as follows.

- If the bidder reports  $v = 5$ , he pays a call option with strike 5 plus 3.75 in cash;
- If the bidder reports  $v = 6$ , he pays a call option with strike 6 plus 4.15 in cash.

- If the bidder reports  $v = 7$ , he pays a call option with strike 7 plus 4.85 in cash.

Expected payments for the

- call option with strike 5 are 0.25 for  $v = 5$ , 1 for  $v = 6$ , and 2 for  $v = 7$ .
- call option with strike 6 are 0 for  $v = 5$ , 0.25 for  $v = 6$ , and 1 for  $v = 7$ .
- call option with strike 7 are 0 for  $v = 5$  and  $v = 6$ , and 0.25 for  $v = 7$ .

Substitution into a bidder's expected profits reveals that truth telling results in zero payoffs and deviating results in strictly negative payoffs. Thus, the mechanism is incentive compatible and all rents are extracted.

### 3 Conclusion

We show that auctions that combine securities with differing steepness may yield higher revenues than using securities of the same steepness when bidders who expect to generate higher revenues from winning the asset also face higher opportunity costs. The advantages of combining arise because the differential rents of high over low bidder types may change sign between sets of securities with differing steepness. Positive mixtures of different sets of securities can balance those differential rents to zero and achieve full surplus extraction.

We provide necessary and sufficient conditions for a seller to be able to extract all rents from bidders by combining cash and equity, both when the cash payment is restricted to be positive, and when cash can be negative. These results are robust: if the conditions for full extraction only approximately hold, our mechanism still extracts almost all surplus. We then show that combining cash with steeper-than-equity securities can extract full rents in settings where cash plus equity can not. Collectively, our findings indicate that a seller should combine the least-steep security (cash) with the steepest security. Tailored to different types, the mix creates wider variation in steepness, which helps to reduce the differential rents of higher types over lower types and to ensure the global incentive compatibility of the mechanism.

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## 5 Appendix

**Proof of Proposition 1:** If bidding is truthful, then the expected profit of a type  $v$  bidder is

$$h(v, v) = y(v) - x(v) + \sup_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')). \quad (33)$$

This expected profit is nonnegative and no more than  $\sup_{v'} (x(v') - y(v')) - \inf_{v'} (x(v') - y(v'))$  for all  $v$ . Thus, individual rationality is satisfied. Next we show truthful bidding is incentive compatible. By (21), (22), (13), (3) and (4), we have

$$q(z) = \frac{dy(z)}{dz} \quad \text{and} \quad \omega(z) = z \frac{dy(z)}{dz} - y(z) - \sup_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')). \quad (34)$$

Then by (5),

$$\begin{aligned} h(v, z) &= \left[ v \frac{dy(z)}{dz} - x(v) \right] - \left[ z \frac{dy(z)}{dz} - y(z) - \sup_{v' \in [\underline{v}, \bar{v}]} (x(v') - y(v')) \right] \\ &= \left[ (v - z) \frac{dy(z)}{dz} - (y(v) - y(z)) \right] + h(v, v), \end{aligned} \quad (35)$$

where (35) follows from (33). Then by the weak convexity of  $y(\cdot)$ ,  $(v - z) \frac{dy(z)}{dz} - (y(v) - y(z)) \leq 0$  regardless of whether  $v \leq z$  or  $v > z$ . Thus, (35) yields

$$h(v, z) - h(v, v) \leq 0,$$

establishing the incentive compatibility of truthful bidding. Furthermore, truthful bidding maximizes social welfare, establishing the result.  $\blacksquare$

**Proof of Lemma 3:** Claim 1:  $ES_v(s, v)$  is increasing in  $s$  for all  $v$ . For  $s_1 > s_2$ , we have

$$ES_v(s_1, v) - ES_v(s_2, v) = \int [s_1(y) - s_2(y)] g_v(y|v) dy.$$

Because  $g_v$  integrates to zero ( $g$  is a probability density), there exists a  $y^*$  such that  $g_v(y^*|v) = 0$ .  $\frac{g_v(y|v)}{g(y|v)}$  is monotone increasing in  $y$  by the sMLRP. Thus, for  $y < y^*$ ,  $\frac{g_v(y|v)}{g(y|v)} < 0$  and hence  $g_v(y|v) < 0$ ; and for  $y > y^*$ ,  $\frac{g_v(y|v)}{g(y|v)} > 0$  and hence  $g_v(y|v) > 0$ . Then since  $s_1(y) - s_2(y)$  is weakly increasing by Assumption 1,

$$\int_{y < y^*} [S(s_1, y) - S(s_2, y)] g_v(y|v) dy \geq \int_{y < y^*} [S(s_1, y^*) - S(s_2, y^*)] g_v(y|v) dy$$

and

$$\int_{y > y^*} [S(s_1, y) - S(s_2, y)] g_v(y|v) dy \geq \int_{y > y^*} [S(s_1, y^*) - S(s_2, y^*)] g_v(y|v) dy.$$

Because  $S(s_1, y) - S(s_2, y)$  strictly increases over a positive measure of  $y$ , one of the inequalities is strict. Adding yields Claim 1:

$$\begin{aligned} ES_v(s_1, v) - ES_v(s_2, v) &= \int_{y < y^*} [S(s_1, y) - S(s_2, y)] g_v(y|v) dy + \int_{y > y^*} [S(s_1, y) - S(s_2, y)] g_v(y|v) dy \\ &> \int_{y < y^*} [S(s_1, y^*) - S(s_2, y^*)] g_v(y|v) dy + \int_{y > y^*} [S(s_1, y^*) - S(s_2, y^*)] g_v(y|v) dy \end{aligned}$$

Lemma 3 then follows from Claim 1 and Assumption 1. ■

**Proof of Theorem 2:** If bidding is truthful, then substituting for  $M(v)$  in (24) reveals that a bidder's equilibrium payoff is zero, regardless of his valuation. To see that truthful bidding is an equilibrium, by Lemma 2, it suffices to establish that  $\frac{\partial h(v, z)}{\partial v} \geq 0$  for  $v < z$  and that  $\frac{\partial h(v, z)}{\partial v} \leq 0$  for  $v > z$ . We have  $\frac{\partial h(v, z)}{\partial v} = l(v, z)$ , where

$$l(v, z) \equiv 1 - x'(v) - ES_v(s(z), v),$$

and (27) yields  $l(z, z) = 0$ . Thus, for  $z \neq v$ ,

$$\begin{aligned} l(v, z) &= l(v, z) - l(z, z) \\ &= x'(z) + ES_v(s(z), z) - x'(v) - ES_v(s(z), v) \\ &= \int_v^z x''(v) dv + \frac{ES_v(s(z), v) - ES_v(s(z), z)}{v - z} (z - v) \\ &= \int_v^z x''(v) dv + \frac{\int_z^v ES_{vv}(s(z), v) dv}{v - z} (z - v) \end{aligned}$$

If  $z > v$ , then by (29)

$$\begin{aligned} l(v, z) &\geq \int_v^z x''(v) dv + k(z)(z - v) \\ &\geq \int_v^z x''(v) dv + K(z - v) = \int_v^z (x''(v) + K) dv \geq 0, \end{aligned}$$

and if  $z < v$ , then

$$\begin{aligned} l(v, z) &\leq \int_v^z x''(v) dv + k(z)(z - v) \\ &\leq \int_v^z x''(v) dv + K(z - v) = \int_v^z (x''(v) + K) dv \leq 0. \end{aligned}$$

Thus,  $\frac{\partial h(v, z)}{\partial v} \geq 0$  for  $v < z$ , and  $\frac{\partial h(v, z)}{\partial v} \leq 0$  for  $v > z$ . Thus,  $h(v, z) \leq 0, \forall z \neq v$ .  $\blacksquare$

**Proof of Corollary 6:** To show that  $M$  in (32) is strictly positive, suppose that  $M \leq 0$ , i.e.,  $ES(s(v), v) \geq v - x(v)$ . Then by Assumption 1 and  $v - x(v) > 0$ , there exists a  $\hat{s} \leq s$  such that  $ES(\hat{s}, v) = v - x(v)$ . The premise of the corollary is that bidding strategies would be strictly decreasing in a second-price security-bid auction. Thus, by  $ES(\hat{s}, v) = v - x(v)$ ,  $ES_v(\hat{s}, v) > 1 - x'(v)$  (Lemma 1, Che and Kim 2010). By  $\hat{s} \leq s$  and Claim 1 in the proof of Lemma 3 ( $ES_v(s, v)$  increases in  $s$ ),  $ES_v(s, v) > 1 - x'(v)$ , contradicting Theorem 2's premise that  $s_i$  solves (27). Thus,  $M > 0$ . Moreover, pure security or pure cash cannot extract full rents because when a seller can combine them, she must use strictly positive amounts of both to extract full rents.  $\blacksquare$

**Proof of Lemma 4:** For notational ease, denote the security's payoff function,  $S(s(v), \cdot)$ , by  $S(\cdot)$ . Then

$$ES(s(v), v) = \int_0^\infty S(\theta v) f_\theta(\theta) d\theta$$

and

$$ES_{vv}(s(v), v) = \int_0^\infty \theta^2 S''(\theta v) f_\theta(\theta) d\theta.$$

Because  $S(\cdot)$  is weakly convex and not a straight line, there exist two values  $v_a$  and  $v_b$ , with  $0 < v_a < v_b < \infty$ , such that  $S'(v_b) - S'(v_a) > 0$ . Define

$$b \equiv \min_{\theta \in (v_a/v, v_b/v)} \theta^2 f_\theta(\theta) > 0.$$

Because  $S'' \geq 0$ ,

$$\begin{aligned} ES_{vv}(s(v), v) &\geq \int_{v_a}^{v_b} \theta^2 S''(\theta v) f_\theta(\theta) d\theta \geq b \int_{\frac{v_a}{v}}^{\frac{v_b}{v}} S''(\theta v) d\theta \\ &= \frac{b}{v} (S'(v_b) - S'(v_a)) > 0. \end{aligned} \quad \blacksquare$$